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The algebra of transition matrices for the $AdS_5 \times S^5$ superstring

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Abstract: We consider integrability properties of the superstring on $AdS_5 \times S^5$ background and construct a new one parameter family of currents which satisfies the vanishing curvature condition. We present the hamiltonian analysis for the sigma model action and determine the Poisson algebra of the transition matrices. We reveal the generalization of the $\mathbb{Z}_4$ automorphism analogous to the sigma models defined on a symmetric space coset. A possible regularization scheme for the ambiguities present, which respects the generalized automorphism, is also discussed.

Keywords: Sigma Models, Superstrings and Heterotic Strings, AdS-CFT and dS-CFT Correspondence, Integrable Field Theories.

Contents

1. Introduction

The AdS/CFT correspondence [1]–[4] has proved to be very crucial in establishing correspondence between superstring and supersymmetric gauge theories. The AdS/CFT correspondence becomes tractable essentially in two different situations. Namely, when the 't Hooft coupling ($g^2_{YM}N$) is small, one can adopt the large N technique to carry out computations in the superYang-Mills (SYM) sector [5]–[11]. On the other hand, for the large values of 't Hooft coupling, the duality argument can be invoked to facilitate computations in the string theory perturbation (inverse tension) frame work [12]–[14]. For values of the 't Hooft coupling which are neither small nor large, there is no obvious simplification. In a recent paper, Bena, Polchinski and Roiban [15] studied the integrability properties of the superstring propagating on the $AdS_5 \times S^5$ background with a goal of exploring the dynamics of the string theory for such an intermediate coupling when the string theory is not necessarily in the perturbative regime.

The integrability properties of SYM have been subject of considerable interest in recent years. The conformal group in four dimensions is identified to be $\mathrm{SO}(4,2)$ and dilatation is one of its generators. It is well known in various computations to establish AdS/CFT correspondence, the Yang-Mills theory lives on $\mathbb{R} \times S^3$ and the dilatation operator D is the hamiltonian in the radial quantization scheme [16]. Minahan and Zarembo [7] computed anomalous dimensions of a class composite operators of $\mathcal{N} = 4$ SYM in the large N limit at the one loop level. They show that the anomalous dimension operator, in the radial quantization, is related to a spin chain quantum hamiltonian which is known to be integrable. There is also correspondence between semiclassical string states and composite operators of SYM [17]. It has been argued by Dolan, Nappi and Witten [18] that the integrability property of SYM unraveled in this context is intimately connected with the yangian symmetry (see for a review [19]) associated with $\mathcal{N}=4$ SYM when one sets the Yang-Mills coupling to zero. Furthermore, they conjecture that the yangian symmetry discovered for the SYM is related to the yangian symmetry which one expects to obtain due to the presence of conserved nonlocal currents for the superstring when it propagates in $AdS_5 \times S^5$ background. Therefore, the hidden symmetries uncovered by Bena, Polchinski and Roiban [15] are connected with the integrability properties of SYM discussed above.

If one considers a bosonic string in an AdS_n space, the worldsheet action may be identified with an $O(n)$ nonlinear σ-model [20] and one may adopt well known prescriptions to construct a family of nonlocal conserved currents [21] for such a case which is responsible for the classical integrability of the model. Let us recall how the $AdS_5 \times S^5$ geometry arises in type IIB string theory. The 5-dimensional extremal black hole solution is obtained with appropriate choice of backgrounds which solve the equations of motion. Of special significance, in this case, is the fact that the five form RR flux and the dilaton assume constant values. Subsequently, the near horizon limit is taken which leads to the geometry of $AdS_5 \times S^5$. Therefore, the worldsheet action for such a theory is to be constructed keeping in mind the presence of constant RR background.

In such a case, the standard technique of [21] becomes inadequate. This is due to the fact that the NSR formalism is inapplicable to construct a suitable nonlinear σ-model action on the worldsheet. Consequently, it is the Green-Schwarz formalism that is more appropriate in this context where the theory can be described as a nonlinear sigma model with a Wess-Zumino-Witten term. In this case, the basic field variables of the action parametrize the coset

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}\,, \tag{1.1}$$

as has been shown by several authors [22]–[24]. It is worth emphasizing here that the Green-Schwarz action for the superstring on $AdS_5 \times S^5$ is not strictly a coset σ-model (unlike the bosonic theories) due the presence of the fermionic Wess-Zumino term as well as the local κ symmetry. The properties of the coset and its grading structure played a key role in construction of the nonlocal currents [15]. Recently, Hatsuda and Yoshida [25] have taken into account the presence of the Wess-Zumino term and constructed explicitly the nonlocal charges as well as the yangian algebra associated with the system. Their work involves a euclideanized supergroup GL(4|4) and the sigma model is defined on a superspace.

In studying the integrability properties of a sigma model, the basic object is a one parameter family of currents which satisfies a zero curvature condition. There is, however, an arbitrariness in the choice of this current. In construction of objects such as the monodromy matrix, it is the current invariant under a generalized inner automorphism of the symmetry group that plays a crucial role [26]. This symmetry is, however, lacking in the one parameter family of currents constructed in [15]. In this paper, therefore, we have tried to study the integrability properties of the superstring on the $AdS_5 \times S^5$ background

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keeping the conventional automorphisms manifest. We construct a one parameter family of currents which manifestly is invariant under the inner automorphism of the graded group $\mathbb{Z}_4$. In spite of the fact that the sigma model in this case is not a genuine coset space model, we find a generalization of the inner automorphism to $\mathbb{Z}_4^\infty$ which is relevant in the construction of the monodromy matrix. We systematically construct the nonlocal charges from this current. Using the conventional sigma model action (in terms of currents) with a Wess-Zumino-Witten term in the coset space [27], we carry out the hamiltonian analysis of the system and determine the basic Poisson brackets of the current (this generates the yangian algebra) which is essential in the construction of the algebra of the transition matrix and which has a closed form. We point out the difficulties that arise due to the presence of the κ symmetry in the action. In addition, we clarify some subtleties associated with the Virasoro constraints of the theory in this case.

The paper is organized as follows. In section 2 we recapitulate the basic properties of the type IIB Green-Schwarz superstring action on the $AdS_5 \times S^5$ background. In section 3 we present essential properties of the superalgebra PSU(2, 2|4). In section 4 we introduce a one parameter family of currents which satisfies the vanishing curvature condition and has a form that reveals the special $\mathbb{Z}_4^\infty$ automorphism. In section 5 we construct the hierarchy of conserved nonlocal currents. In the final section 6 we perform the hamiltonian analysis and calculate the Poisson bracket of the flat currents. We discuss various aspects of the results as well as future directions for this analysis and conclude with a brief summary in section 7.

2. Superstring on $AdS_5 \times S^5$

We summarize here some of the basic properties of the type IIB Green-Schwarz superstring action on the $AdS_5 \times S^5$ background [22, 23, 28, 29]. The superstring can be defined as a non-linear sigma model on the coset superspace

$$\frac{G}{H} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}\,. \tag{2.1}$$

The classical action has the Wess-Zumino-Witten form

$$S = -\frac{1}{2}\int_{\partial M^3} d^2\sigma\sqrt{-g}g^{ij}\left(L_i^{\hat a}L_j^{\hat a}\right) + i\int_{M^3} s^{IJ}\left(L^{\hat a}\wedge\overline{L}^I\gamma^{\hat a}\wedge L^J\right), \tag{2.2}$$

where $g^{ij}, i, j = 0, 1$ represents the worldsheet metric, $s^{IJ} = \mathrm{diag}(1,-1), I, J = (1,2); \hat a = (a, a')$ with $a = (0,\ldots,4)$ and $a' = (5,\ldots,9)$ corresponding to tangent space indices for AdS_5 and S^5 respectively. We use the convention that repeated indices are summed. The supervielbeins $L^{\hat a}$ and L^I are defined as

$$\begin{aligned} L^I &= \left(\left(\frac{\sinh\mathcal{M}}{\mathcal{M}}\right)D\theta\right)^I, \\ L^{\hat a} &= e^{\hat a}_{\hat\mu}(x)\mathrm{d}x^{\hat\mu} - i\overline{\theta}\gamma^{\hat a}\left(\left(\frac{\sinh\mathcal{M}/2}{\mathcal{M}/2}\right)^2 D\theta\right), \end{aligned} \tag{2.3}$$

where

$$\left(\mathcal{M}^2\right)^{IJ} = \epsilon^{IK}\left(-\gamma^a\theta^K\bar{\theta}^J\gamma^a + \gamma^{a'}\theta^K\bar{\theta}^J\gamma^{a'}\right) + \frac{1}{2}\epsilon^{KJ}\left(\gamma^{ab}\theta^I\bar{\theta}^K\gamma^{ab} - \gamma^{a'b'}\theta^I\bar{\theta}^K\gamma^{a'b'}\right). \quad (2.4)$$

Here $(x^{\hat{\mu}}, \theta^I)$ denote the bosonic and fermionic string coordinates in the target space, $(e^{\hat{a}}, \omega^{\hat{a}\hat{b}})$ are the bosonic vielbein and the spin connection respectively and the covariant differential is given by

$$(D\theta)^I = \left[\delta^{IJ}\left(\mathrm{d} + \frac{1}{4}\,\omega^{\hat{a}\hat{b}}\gamma_{\hat{a}\hat{b}}\right) - \frac{i}{2}\epsilon^{IJ}\; e^{\hat{a}}\gamma_{\hat{a}}\right]\theta^J\,. \quad (2.5)$$

The equations of motion following from the action (2.2) take the forms

$$\sqrt{-g}g^{ij}\left(\nabla_i L^a_j + L^{ab}_i L^b_j\right) + i\epsilon^{ij}s^{IJ}\bar{L}^I_i\gamma^a L^J_j = 0\,, \quad (2.6)$$

$$\sqrt{-g}g^{ij}\left(\nabla_i L^{a'}_j + L^{a'b'}_i L^{b'}_j\right) - \epsilon^{ij}s^{IJ}\bar{L}^I_i\gamma^{a'} L^J_j = 0\,, \quad (2.7)$$

$$\left(\gamma^a L^a_i + i\gamma^{a'}L^{a'}_i\right)\left(\sqrt{-g}g^{ij}\delta^{IJ} - \epsilon^{ij}s^{IJ}\right)L^J_j = 0\,, \quad (2.8)$$

with ∇_i representing the covariant derivative on the worldsheet. We will use the conformal gauge $\sqrt{-g}g^{ij} = \eta^{ij}$ in which case the equations of motion (2.6)–(2.8) simplify, but should be complemented with the Virasoro constraints

$$L^{\hat{a}}_i L^{\hat{a}}_j = \frac{1}{2}\,\eta_{ij}\eta^{kl}\; L^{\hat{a}}_k L^{\hat{a}}_l\,. \quad (2.9)$$

To consider the integrability properties of the sigma model we will need some properties of the superalgebra PSU$(2,2|4)$ which we review in the next section.

3. Properties of PSU$(2,2|4)$

In this section we discuss some of the essential properties of the superalgebra PSU$(2,2|4)$ [27] and [30]–[32]. Since we are interested in a supersymmetric field theory, we assume that the algebra is defined on a Grassmann space, PSU$(2,2|4;\mathbb{C}B_L)$. We represent an element of this superalgebra by an even supermatrix of the form

$$G = \begin{pmatrix} A & X \\ Y & B \end{pmatrix}, \quad (3.1)$$

where A and B are matrices with Grassmann even functions while X and Y are those with Grassmann odd functions, each representing a 4×4 matrix. (An odd supermatrix, on the other hand, has the same form, with A and B consisting of Grassmann odd functions while $X$and Y consisting of Grassmann even functions.)

An element G (see 3.1) of the superalgebra PSU$(2,2|4;\mathbb{C}B_L)$ is given by a 8×8 matrix, satisfying

$$GK + KG^{\ddagger} = 0\,, \quad (3.2)$$

$$\mathrm{tr}A = \mathrm{tr}B = 0\,, \quad (3.3)$$

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where $K = \begin{pmatrix} \Sigma & 0 \\ 0 & I_4 \end{pmatrix}$ and $\Sigma = \sigma_3 \otimes I_2$ with I_2, I_4 representing the identity matrix in 2 and 4 dimensions respectively. The $\ddagger$ is defined by

$$G^{\ddagger} = G^{\mathrm{T}\sharp}\,, \tag{3.4}$$

where T denotes transposition and $\sharp$ is a generalization of complex conjugation which acts on the functions c of the matrices as

$$c^{\sharp} = \begin{cases} c^* & (\text{for } c \text{ Grassmann even}) \\ -ic^* & (\text{for } c \text{ Grassmann odd}) \end{cases}\,. \tag{3.5}$$

The condition (3.2) can be written explicitly as

$$\Sigma A^{\dagger} + A\Sigma = 0\,, \qquad B^{\dagger} + B = 0\,, \qquad X - i\Sigma Y^{\dagger} = 0\,. \tag{3.6}$$

The essential feature of the superalgebra PSU$(2,2|4)$ is that it admits a $\mathbb{Z}_4$ automorphism such that the condition $\mathbb{Z}_4(H) = H$ determines the maximal subgroup to be $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ which leads to the definition of the coset for the sigma model. (This is the generalization of the $\mathbb{Z}_2$ automorphism of bosonic sigma models to the supersymmetric case.) The $\mathbb{Z}_4$ automorphism Ω takes an element of PSU$(2,2|4)$ to another, $G \to \Omega(G)$, such that

$$\Omega(G) = \begin{pmatrix} JA^{\mathrm{T}}J & -JY^{\mathrm{T}}J \\ JX^{\mathrm{T}}J & JB^{\mathrm{T}}J \end{pmatrix}, \tag{3.7}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It follows now that $\Omega^4(G) = G$.

Since $\Omega^4 = 1$, the eigenvalues of Ω are i^p with $p = 0,1,2,3$. Therefore, we can decompose the superalgebra as

$$G = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3\,, \tag{3.8}$$

where $\mathcal{H}_p$ denotes the eigenspace of Ω such that if $H_p \in \mathcal{H}_p$, then

$$\Omega(H_p) = i^p H_p\,. \tag{3.9}$$

We have already noted that $\Omega(\mathcal{H}_0) = \mathcal{H}_0$ determines $\mathcal{H}_0 = \mathrm{SO}(4,1) \times \mathrm{SO}(5)$. $\mathcal{H}_2$ represents the remaining bosonic generators of the superalgebra while $\mathcal{H}_1, \mathcal{H}_3$ consist of the fermionic generators of the algebra. (In a bosonic sigma model, $\mathcal{H}_0, \mathcal{H}_2$ are represented respectively as Q, P.) The automorphism also implies that

$$[H_p, H_q] \in \mathcal{H}_{p+q \pmod 4}\,. \tag{3.10}$$

The space $\mathcal{H}_p$ is spanned by the generators $(t_p)_A$ of the superalgebra so that we can explicitly write

$$\begin{aligned} G &= (H_p)^A (t_p)_A \\ &= (H_0)^m (t_0)_m + (H_1)^{\alpha_1}(t_1)_{\alpha_1} + (H_2)^a (t_2)_a + (H_3)^{\alpha_2}(t_3)_{\alpha_2}\,, \end{aligned} \tag{3.11}$$

where $A = (m, \alpha_1, a, \alpha_2)$ take values over all the generators of the superalgebra, $(H_0)^m$ and $(H_2)^a$ are Grassmann even functions, while $(H_1)^{\alpha 1}$ and $(H_3)^{\alpha 2}$ are Grassmann odd functions. The generators satisfy the graded algebra PSU$(2,2|4)$,

$$[(t_p)_A, (t_q)_B] = f_{AB}{}^C (t_{p+q})_C\,, \tag{3.12}$$

where $p+q$ on the right hand side is to be understood modulo 4.

The Killing form (or the bilinear form) $\langle H_p, H_q\rangle$ is also $\mathbb{Z}_4$ invariant so that

$$\langle \Omega(H_p), \Omega(H_q)\rangle = \langle H_p, H_q\rangle\,. \tag{3.13}$$

This implies that

$$i^{(p+q)}\langle H_p, H_q\rangle = \langle H_p, H_q\rangle\,, \tag{3.14}$$

which leads to

$$\langle H_p, H_q\rangle = 0 \quad \text{unless} \quad p+q=0 \quad (\text{mod } 4). \tag{3.15}$$

Since the supertrace of a supermatrix M is defined as

$$\mathrm{str}(M) = \begin{cases} \mathrm{tr}A - \mathrm{tr}B \text{ (if } M \text{ is an even supermatrix)} \\ \mathrm{tr}A + \mathrm{tr}B \text{ (if } M \text{ is an odd supermatrix)} \end{cases}, \tag{3.16}$$

and the metric of the algebra is defined as $G_{AB} = \mathrm{str}((t_p)_A (t_q)_B)$, the above relation also implies that only the components $G_{mn}, G_{ab}, G_{\alpha_1\alpha_2} = -G_{\alpha_2\alpha_1}$ of the metric are non-zero. The structure constants possess the graded anti-symmetry property

$$f^D_{AB} G_{DC} = -(-)^{|A||B|} f^D_{BA} G_{DC} = -(-)^{|B||C|} f^D_{AC} G_{DB}\,, \tag{3.17}$$

where $|A|$ denotes the Garssmann parity of A, namely, $|A|$ is 0 when A is m or a, while $|A|$ is 1 when A is α_1 or α_2.

4. The flat current

Let us consider the map g from the string worldsheet into the graded group PSU$(2,2|4)$. In this case, the current 1 form $J = -g^{-1}\mathrm{d}g$ belongs to the superalgebra and therefore can be decomposed as

$$J = -g^{-1}\mathrm{d}g = H + P + Q^1 + Q^2\,, \tag{4.1}$$

where, in terms of our earlier notation, we can identify

$$H = H_0\,, \qquad Q^1 = H_1\,, \qquad P = H_2\,, \qquad Q^2 = H_3\,. \tag{4.2}$$

From the definition of the current in (4.1), we see that it satisfies a zero curvature condition

$$\mathrm{d}J - J \wedge J = 0\,. \tag{4.3}$$

In terms of the components of the current (4.2), the equations of motion can be written as [15]

$$\mathrm{d}\,{}^*P = {}^*P \wedge H + H \wedge {}^*P + \frac{1}{2}\left(Q \wedge Q' + Q' \wedge Q\right),$$

$$
\begin{aligned}
0 &= P \wedge ({}^*Q - Q') + ({}^*Q - Q') \wedge P\,, \\
0 &= P \wedge (Q - {}^*Q') + (Q - {}^*Q') \wedge P\,,
\end{aligned} \tag{4.4}
$$

where * denotes the Hodge star operation and we have defined $Q \equiv Q^1 + Q^2, Q' \equiv Q^1 - Q^2$.

Let us next introduce a one parameter family of currents $\hat{J}(t) \equiv -\hat{g}^{-1}(t)\mathrm{d}\hat{g}(t)$ where t is a constant spectral parameter (here we are suppressing the dependence on the worldsheet coordinates)

$$
\hat{J}(t) = H + \frac{1+t^2}{1-t^2}\,P + \frac{2t}{1-t^2}\,{}^*P + \sqrt{\frac{1}{1-t^2}}\,Q + \sqrt{\frac{t^2}{1-t^2}}\,Q'\,, \tag{4.5}
$$

such that $\hat{J}(t=0) = J$. It is easy to check that the vanishing curvature condition for this new current

$$
\mathrm{d}\hat{J} - \hat{J} \wedge \hat{J} = 0\,, \tag{4.6}
$$

leads to all the equations of motion (4.4) as well as the zero curvature condition (4.3).

It is worth comparing the form of this one parameter family of currents (4.5) with that for a (bosonic) two dimensional sigma model obtained from the string effective action dimensionally reduced from D-dimensions [33]. In this case, the sigma model coupled to gravity is defined on the symmetric space $\frac{G}{H} = \frac{O(d,d)}{O(d)\times O(d)}$ where $d = D-2$ and the corresponding one parameter family of currents has the form (conventionally in a bosonic sigma model $\hat{g}$ is written as $\hat{V}$, but we are using the same letter for ease of comparison)

$$
\hat{J}(t) = \hat{g}^{-1}(t)\mathrm{d}\hat{g}(t) = H + \frac{1+t^2}{1-t^2}\,P + \frac{2t}{1-t^2}\,{}^*P\,, \tag{4.7}
$$

where the components of the currents belong to the appropriate spaces. This has the same form as (4.5) if we set the fermionic generators to zero (the groups are, of course, different). However, apart from the presence of the fermionic degrees of freedom in the $AdS_5 \times S^5$ theory, the essential difference between the two theories lies in the fact that when we dimensionally reduce the string effective action to two space-time dimensions, it describes a sigma model coupled to gravity. As is well known, in such a theory the spectral parameter assumes space-time dependence for consistency. Namely, the consistent zero curvature description in this case requires that the spectral parameter satisfies the condition

$$
\partial_\alpha \rho = -\frac{1}{2}\varepsilon_{\alpha\beta}\partial^\beta\left(e^{-\phi}\left(t + \frac{1}{t}\right)\right), \tag{4.8}
$$

where ϕ is the shifted dilaton and $\rho = e^{-\phi}$. In contrast, in the case of $AdS_5 \times S^5$, the spectral parameter is a global parameter.

There is, however, an important symmetry which the one parameter family of currents in both the theories share. In the bosonic sigma model coupled to gravity, the theory is invariant under a generalization of the symmetric space automorphism (which is $\mathbb{Z}_2$ as we have alluded to earlier)

$$
\eta^\infty\left(\hat{g}(t)\right) = \left(\hat{g}^{-1}\left(\frac{1}{t}\right)\right)^{\mathrm{T}}. \tag{4.9}
$$

This symmetry is essential in the construction of the monodromy matrix $\mathcal{M} = \hat{g}(t)\hat{g}^{\mathrm{T}}(\frac{1}{t})$ [26, 34] which encodes integrability properties and is related to the transition matrix [35, 36]. In the case of the superstring on the $AdS_5 \times S^5$ background, the coset (2.1) is not exactly a symmetric space. We can, however, still define a generalization of the $\mathbb{Z}_4$ automorphism of the superalgebra to $\mathbb{Z}_4^\infty$ in a manner analogous to (4.9). Explicitly, it is easy to check that the one parameter family of currents is invariant under (we choose $\sqrt{-1} = i$)

$$\begin{aligned} t &\longrightarrow \frac{1}{t}\,, \\ H &\longrightarrow H\,, \qquad P \longrightarrow -P\,, \\ Q &\longrightarrow -iQ', \quad Q' \longrightarrow -iQ\,. \end{aligned} \tag{4.10}$$

Consequently, this current is different from the one constructed in [15] which does not have the necessary invariance property under the inner automorphism. (There is an arbitrariness in the choice of the one parameter family of flat currents and we have constructed one that has the desired behavior under the inner automorphism of the symmetry algebra which parallels the construction in conventional sigma models.) This symmetry will allow us to define the monodromy matrix for the present case in a way similar to the symmetric space bosonic sigma model [34] and is, consequently, quite important. We note that one of the fermionic constraints in (4.4) is, in fact, a consequence of this $\mathbb{Z}_4^\infty$ symmetry.

5. Nonlocal currents

The sigma model action which leads to the equations of motion (4.4) has the form

$$S = \frac{1}{2}\int \mathrm{str}\left(P \wedge {}^*P - Q^1 \wedge Q^2\right), \tag{5.1}$$

where P, Q^1, Q^2 are defined in (4.1). The first term in the action (5.1) is similar to that in the principal chiral model, while the second represents the WZW term [27]. This action is manifestly invariant under a local (gauge) transformation $g \to gh$ where $h \in \mathrm{SO}(4,1) \times \mathrm{SO}(5)$ is a local function, since under such a transformation

$$\begin{aligned} H &\longrightarrow h^{-1}H\,h - h^{-1}\mathrm{d}h\,, \\ P &\longrightarrow h^{-1}P\,h\,, \\ Q^{1,2} &\longrightarrow h^{-1}Q^{1,2}\,h\,. \end{aligned} \tag{5.2}$$

The action is also invariant under a global transformation (left multiplication) $g \to \omega g$ where $\omega \in \mathrm{PSU}(2,2|4)$ and the corresponding conserved Noether current for this left translation can be obtained from the action (5.1) to be

$$j^{(0)} = p + \frac{1}{2}{}^*q', \quad \mathrm{d}\,{}^*j^{(0)} = 0\,. \tag{5.3}$$

(The meaning of the superscript will become clear shortly, namely, it is the zeroth order current in the infinite hierarchy of conserved currents.) Here we have adopted the notations of [15] for the left and right invariant currents and the relation

$$x = gXg^{-1}\,, \tag{5.4}$$

where lower and upper case objects denote left and right invariant quantities respectively.

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We can now determine the conserved non-local charges associated with the system easily following [38] and [39, 40]. Let us note that the covariant derivative

$$\hat{D}_\mu \equiv \partial_\mu - \hat{J}_\mu(t)\,, \tag{5.5}$$

satisfies the zero curvature condition

$$[\hat{D}_\mu, \hat{D}_\nu] = 0\,. \tag{5.6}$$

As a result, the equation

$$\hat{D}_\mu \chi = \partial_\mu \chi - \hat{J}_\mu(t)\chi = 0\,, \tag{5.7}$$

is integrable. It is convenient to rewrite (5.7) in the following equivalent form

$$\varepsilon_{\mu\nu}\partial^\nu \chi = \varepsilon_{\mu\nu}\hat{J}^\nu(t)\chi - t\hat{J}_\mu(t)\chi + t\partial_\mu\chi\,. \tag{5.8}$$

Using the form of $\hat{J}_\mu(t)$ given in (4.5), we can write (5.8) as

$$\begin{aligned}&\varepsilon_{\mu\nu}\left(\partial^\nu - H^\nu - P^\nu - Q^\nu\right)\chi = \qquad\qquad (5.9)\\ &= t\left(\partial_\mu - H_\mu + P_\mu - \frac{1}{\sqrt{1-t^2}}Q_\mu - \frac{t}{\sqrt{1-t^2}}\left(Q'\right)_\mu + \frac{1}{\sqrt{1-t^2}}\varepsilon_{\mu\nu}\left(Q'\right)^\nu + S(t)\varepsilon_{\mu\nu}Q^\nu\right)\chi,\end{aligned}$$

where $S(t)$ can be determined from

$$\frac{1}{\sqrt{1-t^2}} = 1 + tS(t)\,. \tag{5.10}$$

Let us note that

$$H^\mu + P^\mu + Q^\mu = -\hat{g}^{-1}(0)\partial^\mu \hat{g}(0) = g^{-1}\partial^\mu g = J^\mu(0)\,, \tag{5.11}$$

so that we can write (5.10) in the form

$$\begin{aligned}&\varepsilon_{\mu\nu}g^{-1}\partial^\nu\left(g\chi\right) = \qquad\qquad (5.12)\\ &= t\left(\partial_\mu - H_\mu + P_\mu - \frac{1}{\sqrt{1-t^2}}Q_\mu - \frac{t}{\sqrt{1-t^2}}\left(Q'\right)_\mu + \frac{1}{\sqrt{1-t^2}}\varepsilon_{\mu\nu}\left(Q'\right)^\nu + S(t)\varepsilon_{\mu\nu}Q^\nu\right)\chi\,.\end{aligned}$$

We can now expand χ in a power series in the spectral parameter as

$$\chi = \sum_{n=0}^{\infty} t^n \chi^{(n-1)}\,. \tag{5.13}$$

Substituting this into (5.13), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} t^n \varepsilon_{\mu\nu}g^{-1}\partial^\nu\left(g\chi^{(n-1)}\right) = \sum_{n=0}^{\infty} t^{(n+1)}\Big\{&\partial_\mu - H_\mu + P_\mu - \frac{1}{\sqrt{1-t^2}}Q_\mu - \frac{t}{\sqrt{1-t^2}}\left(Q'\right)_\mu +\\ &+ \frac{1}{\sqrt{1-t^2}}\varepsilon_{\mu\nu}\left(Q'\right)^\nu + S(t)\varepsilon_{\mu\nu}Q^\nu\Big\}\chi^{(n-1)}\,.\end{aligned} \tag{5.14}$$

Furthermore, expanding $\frac{1}{\sqrt{1-t^2}}$ as a power series

$$\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots = \sum_n r_n t^n\,, \tag{5.15}$$

we can iteratively determine all the $\chi^{(n)}$'s from (5.14).

We note that the lowest order term in powers of t (namely, t^0) in (5.14) gives

$$\varepsilon_{\mu\nu} g^{-1}\partial^\nu \left(g\chi^{(-1)}\right) = 0 \tag{5.16}$$

$$\text{or,} \qquad g^{-1}\partial^\nu g\chi^{(-1)} + \partial^\nu \chi^{(-1)} = 0\,. \tag{5.17}$$

This implies that $g\chi^{(-1)}$ is a constant and it is convenient to choose

$$g\chi^{(-1)} = \frac{1}{2}\,, \tag{5.18}$$

for later purposes. In the linear order in t, Eq. (5.14) leads to (after using (5.17))

$$\varepsilon_{\mu\nu} g^{-1}\partial^\nu \left(g\chi^{(0)}\right) = 2\left(P_\mu + \frac{1}{2}\varepsilon_{\mu\nu}(Q')^\nu\right)\chi^{(-1)} \tag{5.19}$$

$$= \left(P_\mu + \frac{1}{2}\varepsilon_{\mu\nu}(Q')^\nu\right) g^{-1}\,, \tag{5.20}$$

and so on.

We can now define the conserved currents associated with the theory as

$$j_\mu^{(n)} = \varepsilon_{\mu\nu}\partial^\nu \left(g\chi^{(n)}\right). \tag{5.21}$$

This leads to the identification

$$\begin{aligned} j_\mu^{(-1)} &= \varepsilon_{\mu\nu}\partial^\nu \left(g\chi^{(-1)}\right) = 0, \\ j_\mu^{(0)} &= \varepsilon_{\mu\nu}\partial^\nu \left(g\chi^{(0)}\right) = g\left(P_\mu + \frac{1}{2}\varepsilon_{\mu\nu}(Q')^\nu\right) g^{-1}\,. \end{aligned} \tag{5.22}$$

Upon using (5.4), the second current takes the form

$$j_\mu^{(0)} = p_\mu + \frac{1}{2}\varepsilon_{\mu\nu}(q')^\nu\,. \tag{5.23}$$

This is precisely the Noether current (5.3) determined earlier which is conserved. Similarly, in the second order in t, we obtain

$$\begin{aligned} j_\mu^{(1)} &= \varepsilon_{\mu\nu}\partial^\nu \left(g\chi^{(1)}\right) \\ &= \varepsilon_{\mu\nu}\left(j^{(0)}\right)^\nu + 2j_\mu^{(0)}\left[g\chi^{(0)}\right] + \frac{1}{4}\varepsilon_{\mu\nu}q^\nu - \frac{1}{2}q_\mu^{'}\,, \end{aligned} \tag{5.24}$$

which can also be written in an explicitly non-local form using (5.22) as

$$j_\mu^{(1)} = \varepsilon_{\mu\nu}\left(j^{(0)}\right)^\nu + 2j_\mu^{(0)}\left(\partial^{-1}j_0^{(0)}\right) + \frac{1}{4}\varepsilon_{\mu\nu}q^\nu - \frac{1}{2}q_\mu^{'}\,. \tag{5.25}$$

It is easy to check (using the form (5.24)) that this current is indeed conserved, $\partial^\mu j_\mu^{(1)} = 0$. Using this iterative procedure we can construct all the conserved currents in the hierarchy which are left invariant. The corresponding algebra of the non-local charges is expected to satisfy a yangian algebra [19, 25].

6. The hamiltonian analysis

The calculation of the algebra of charges can be carried out once the basic Poisson brackets of the theory have been determined. Basically, we are interested in the Poisson algebra of the transition matrices. Let us recall that the transition matrix $T(\sigma_1, \sigma_2, t)$ is defined using the current (4.5) which satisfies the zero curvature condition as

$$T(\sigma_1, \sigma_2, t) = g^{-1}(\sigma_1, t) g(\sigma_2, t) = \mathrm{P}\left(e^{\int_{\sigma_2}^{\sigma_1} \mathrm{d}\sigma \ \hat{J}_1(\sigma, t)}\right). \tag{6.1}$$

Here, we have put back the explicit dependence on the worldsheet coordinates whose spatial component is denoted by σ. (The transition matrix is simply an open Wilson line along a spatial path. The spatial coordinate σ is periodic for the $AdS_5 \times S^5$ string and this causes some technicalities in defining the charges. We avoid such questions by working directly with the transition matrix.) It follows now [41, 42] that

$$\begin{aligned}\{\overset{1}{T}(\sigma_1, \sigma_2, t_1), \overset{2}{T}(\sigma_1', \sigma_2', t_2)\} = &\int_{\sigma_2}^{\sigma_1} \mathrm{d}\sigma \int_{\sigma_2'}^{\sigma_1'} \mathrm{d}\sigma' \left(\overset{1}{T}(\sigma_1, \sigma, t_1), \overset{2}{T}(\sigma_1', \sigma', t_2)\right) \times \\ &\times \left\{\overset{1}{\hat{J}}_1(\sigma, t_1), \overset{2}{\hat{J}}_1(\sigma', t_2)\right\} \left(\overset{1}{T}(\sigma, \sigma_2, t_1), \overset{2}{T}(\sigma', \sigma_2', t_2)\right)\end{aligned} \tag{6.2}$$

We note here that (6.2) represents a matrix Poisson bracket written in an index free tensor notation [41] (which we will follow in our analysis) defined as

$$\overset{1}{A} = A \otimes I\,, \qquad \overset{2}{B} = I \otimes B, \tag{6.3}$$

where

$$(A \otimes B)_{ij,km} = A_{ik} B_{jm}\,. \tag{6.4}$$

It follows from (6.3) and (6.4) that

$$(A \otimes B)(C \otimes D) = (-)^{\epsilon_B \epsilon_C} (AC \otimes BD)\,. \tag{6.5}$$

Furthermore, from (6.2), we see that in order to compute the Poisson bracket between the transition matrix, it is necessary to evaluate $\{\overset{1}{\hat{J}}_1(\sigma, t_1), \overset{2}{\hat{J}}_1(\sigma', t_2)\}$ which we can do only after carrying out a hamiltonian analysis of the model.

The hamiltonian analysis [37] can be carried out starting from the action (5.1) in a way similar to [36]. We treat the space component of the current J_μ as the dynamical variable. The zero curvature condition (4.3) then allows us to determine the time component of the current as

$$J_0 = D^{-1}(\partial_0 J_1)\,, \tag{6.6}$$

where (we are identifying $\partial_0 = \frac{\partial}{\partial \tau}, \partial_1 = \frac{\partial}{\partial \sigma}$ corresponding to the two worldsheet coordinates)

$$D = \partial_1 - [J_1, \cdot]\,. \tag{6.7}$$

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The canonical momentum can now be obtained from an arbitrary variation of the action (5.1) satisfying (6.6) and leads to (we use the left derivatives for fermionic degrees of freedom)

$$\Pi_J \equiv \Pi_H \oplus \Pi_{Q^1} \oplus \Pi_P \oplus \Pi_{Q^2} = -D^{-1}\left(P_0 + \frac{1}{2}Q_1^1 - \frac{1}{2}Q_1^2\right). \tag{6.8}$$

In components (see section 3), the basic canonical Poisson bracket structures are given by (at equal time)

$$\{P_1^a(\sigma), (\Pi_P)_b(\sigma')\} = \delta_b^a \delta(\sigma - \sigma'), \tag{6.9}$$

$$\{H_1^m(\sigma), (\Pi_H)_n(\sigma')\} = \delta_n^m \delta(\sigma - \sigma'), \tag{6.10}$$

$$\{(Q_1^1)^{\alpha_1}(\sigma), (\Pi_{Q^2})_{\beta_1}(\sigma')\} = -\delta_{\beta_1}^{\alpha_1} \delta(\sigma - \sigma'), \tag{6.11}$$

$$\{(Q_1^2)^{\alpha_2}(\sigma), (\Pi_{Q^1})_{\beta_2}(\sigma')\} = -\delta_{\beta_2}^{\alpha_2} \delta(\sigma - \sigma'). \tag{6.12}$$

The extra minus sign in (6.11) and (6.12) arises as a result of the definition of the generalized Poisson brackets involving fermionic systems [37]

$$\{F, G\} = \left[\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial \pi_i} - \frac{\partial F}{\partial \pi_i}\frac{\partial G}{\partial q^i}\right] + (-)^{\epsilon_F}\left[\frac{\partial^L F}{\partial \theta^\alpha}\frac{\partial^L G}{\partial \pi_\alpha} + \frac{\partial^L F}{\partial \pi_\alpha}\frac{\partial^L G}{\partial \theta^\alpha}\right], \tag{6.13}$$

where the superscript L represents left derivation while ϵ_F denotes Grassmann parity of F.

Decomposing relation (6.8) into the appropriate subspaces, we obtain

$$P_0 = -\partial_1 \Pi_P + [H_1, \Pi_P] + [P_1, \Pi_H] + [Q_1^1, \Pi_{Q^1}] + [Q_1^2, \Pi_{Q^2}], \tag{6.14}$$

$$\varphi_1 = -\partial_1 \Pi_H + [H_1, \Pi_H] + [P_1, \Pi_P] + [Q_1^1, \Pi_{Q^2}] + [Q_1^2, \Pi_{Q^1}] \approx 0, \tag{6.15}$$

$$\varphi_2 = -\frac{1}{2}Q_1^1 - \partial_1 \Pi_{Q^1} + [H_1, \Pi_{Q^1}] + [P_1, \Pi_{Q^2}] + [Q_1^1, \Pi_H] + [Q_1^2, \Pi_P] \approx 0, \tag{6.16}$$

$$\varphi_3 = \frac{1}{2}Q_1^2 - \partial_1 \Pi_{Q^2} + [H_1, \Pi_{Q^2}] + [P_1, \Pi_{Q^1}] + [Q_1^1, \Pi_P] + [Q_1^2, \Pi_H] \approx 0. \tag{6.17}$$

Since P_0 contains time derivatives (see (6.6)), the first relation can be used to express velocities in terms of canonical momenta. The last three relations, on the other hand, define primary constraints of the theory. In particular, the first constraint (6.15) is the generator of gauge transformation (5.2). On the other hand, as we will see the last two fermionic constraints give rise to the κ-symmetry. The three primary constraints (6.15), (6.16) and (6.17) should be supplemented with the standard Virasoro constraints, which in our notation take the forms

$$\varphi_4 = \frac{1}{2}\,\mathrm{str}(P_0^2 + P_1^2) \approx 0, \tag{6.18}$$

$$\varphi_5 = \mathrm{str}(P_0 P_1) \approx 0, \tag{6.19}$$

and can also be written as

$$\varphi_\pm = \mathrm{str}\,(P_0 \pm P_1)^2 \approx 0. \tag{6.20}$$

The Poisson brackets (6.9), (6.10), (6.11) and (6.12) can be written in the index free tensor notation as

$$\left\{\overset{1}{P}_1(\sigma), \overset{2}{\Pi}_P(\sigma')\right\} = \Omega_P \delta(\sigma - \sigma'),$$

$$
\begin{aligned}
\left\{\overset{1}{H}_1(\sigma),\overset{2}{\Pi}_H(\sigma')\right\} &= \Omega_H\delta(\sigma-\sigma'),\\
\left\{\overset{1}{Q}{}^1_1(\sigma),\overset{2}{\Pi}_{Q^2}(\sigma')\right\} &= \Omega_{Q^{12}}\delta(\sigma-\sigma')\,,\\
\left\{\overset{1}{Q}{}^2_1(\sigma),\overset{2}{\Pi}_{Q^1}(\sigma')\right\} &= \Omega_{Q^{21}}\delta(\sigma-\sigma')\,,
\end{aligned}
\tag{6.21}
$$

where we have introduced the Casimir operators

$$
\begin{aligned}
\Omega_P &= t_p\otimes t_{p'}G^{pp'}\,, \qquad \Omega_H = t_h\otimes t_{h'}G^{hh'}\,,\\
\Omega_{Q^{12}} &= t_{\alpha_1}\otimes t_{\alpha_2}G^{\alpha_1\alpha_2}\,, \qquad \Omega_{Q^{21}} = t_{\alpha_2}\otimes t_{\alpha_1}G^{\alpha_2\alpha_1}\,.
\end{aligned}
\tag{6.22}
$$

It can now be derived in a straightforward manner (using the relations (A.1) presented in the appendix) that

$$
\begin{aligned}
\left\{\overset{1}{P}_0(\sigma),\overset{2}{H}_1(\sigma')\right\} &= -\left[\Omega_P,\overset{2}{P}_1(\sigma)\right]\delta(\sigma-\sigma'),\\
\left\{\overset{1}{P}_0(\sigma),\overset{2}{Q}{}^1_1(\sigma')\right\} &= -\left[\Omega_P,\overset{2}{Q}{}^2_1(\sigma)\right]\delta(\sigma-\sigma')\,,\\
\left\{\overset{1}{P}_0(\sigma),\overset{2}{Q}{}^2_1(\sigma')\right\} &= -\left[\Omega_P,\overset{2}{Q}{}^1_1(\sigma)\right]\delta(\sigma-\sigma')\,,\\
\left\{\overset{1}{P}_0(\sigma),\overset{2}{P}_1(\sigma')\right\} &= -\left[\Omega_P,\overset{2}{H}_1(\sigma)\right]\delta(\sigma-\sigma')+\Omega_P\partial_\sigma\delta(\sigma-\sigma')\,,\\
\left\{\overset{1}{P}_0(\sigma),\overset{2}{P}_0(\sigma')\right\} &= -\left[\Omega_P,\overset{2}{\varphi}_1(\sigma)\right]\delta(\sigma-\sigma')\,.
\end{aligned}
\tag{6.23}
$$

We note that there is a non-ultra local term in the Poisson brackets (6.23) (namely, the term with the derivative acting on the delta function which in the context of integrable systems is called a non-ultralocal term, while it is known as a Schwinger term in field theory. We will use these two terms interchangeably.). It is well known that the presence of such a term leads to problems in the calculation of the Poisson brackets between the transition matrices. In the bosonic case considered in [36], this problematic term is naturally regularized in the calculation of the algebra of transition matrices in the presence of the dilaton field. When there is no coupling to gravity (and hence a constant spectral parameter), there are also several methods of regularizing the calculations and we will discuss some of them later. We note that the presence of κ symmetry, in the present case, also complicates the calculations. The next step in our analysis is, therefore, to determine all the secondary constraints of the theory and group the constraints into first class and second class constraints. The total hamiltonian density of the theory is easily determined to be

$$
\mathcal{H}_T = \mathrm{str}\left(\frac{1}{2}\left(P_0^2+P_1^2\right)+\lambda_1\varphi_1+\lambda_2\varphi_2+\lambda_3\varphi_3\right), \tag{6.24}
$$

where λ_1, λ_2 and λ_3 denote the Lagrange multipliers corresponding to the three primary constraints. φ_1 is easily checked to be stationary while requiring the constraints φ_2, φ_3 to be stationary determines two of the Lagrange multipliers to correspond to

$$\lambda_2 = -Q_1^2, \quad \lambda_3 = Q_1^1 \,. \tag{6.25}$$

There are no additional secondary constraints and the Lagrange multiplier λ_1 is undetermined corresponding to the fact that φ_1 is the generator of a gauge symmetry. φ_4, φ_5 or equivalently $\varphi_\pm$ can also be checked to be conserved under the hamiltonian flow. It is important to emphasize here that unlike in bosonic models where the standard Virasoro constraints correspond to first class constraints representing generators of reparameterization transformations, in the present theory (in this formulation) with fermions, this is not true. On the other hand, one can define a linear combination of the constraints as

$$\begin{aligned} \bar{\varphi}_4 &= \varphi_4 + \mathrm{str}\ (\lambda_2\varphi_2 + \lambda_3\varphi_3) \approx 0\,, \\ \bar{\varphi}_5 &= \varphi_5 - \mathrm{str}\ (\lambda_2\varphi_2 - \lambda_3\varphi_3) \approx 0\,, \end{aligned} \tag{6.26}$$

which can be easily checked to correspond to first class constraints and generate the reparameterization transformation (of course, $\bar{\varphi}_4$ corresponds to the hamiltonian).

It is straightforward to calculate the Poisson brackets between the constraints

$$\begin{aligned} \left\{\overset{1}{\varphi}_1(\sigma), \overset{2}{\varphi}_1(\sigma')\right\} &= -\left[\Omega_H, \overset{2}{\varphi}_1(\sigma)\right]\delta(\sigma-\sigma') \approx 0\,, \\ \left\{\overset{1}{\varphi}_1(\sigma), \overset{2}{\varphi}_2(\sigma')\right\} &= -\left[\Omega_H, \overset{2}{\varphi}_2(\sigma)\right]\delta(\sigma-\sigma') \approx 0\,, \\ \left\{\overset{1}{\varphi}_1(\sigma), \overset{2}{\varphi}_3(\sigma')\right\} &= -\left[\Omega_H, \overset{2}{\varphi}_3(\sigma)\right]\delta(\sigma-\sigma') \approx 0\,. \end{aligned} \tag{6.27}$$

This shows explicitly that φ_1 is indeed a first class constraint as we expect. The algebra between φ_2 and φ_3, on the other hand, is more complicated

$$\left\{\overset{1}{\varphi}_2(\sigma), \overset{2}{\varphi}_2(\sigma')\right\} = -\left[\Omega_{Q^{12}}, (P_0 \overset{(2)}{-} P_1)(\sigma)\right]\delta(\sigma-\sigma')\,, \tag{6.28}$$

$$\left\{\overset{1}{\varphi}_2(\sigma), \overset{2}{\varphi}_3(\sigma')\right\} = -\left[\Omega_{Q^{12}}, \overset{2}{\varphi}_1(\sigma)\right]\delta(\sigma-\sigma') \approx 0\,, \tag{6.29}$$

$$\left\{\overset{1}{\varphi}_3(\sigma), \overset{2}{\varphi}_3(\sigma')\right\} = -\left[\Omega_{Q^{21}}, (P_0 \overset{(2)}{+} P_1)(\sigma)\right]\delta(\sigma-\sigma')\,. \tag{6.30}$$

We see from (6.28) and (6.30) that φ_2 and φ_3 define a non trivial algebra. These constraints are, however, reducible because of the constraints (6.18) and (6.19). One, therefore, has to further decompose φ_2 and φ_3 into first and second class constraints using some relevant projection [25]. The resulting first class constraints will then generate the κ-symmetry. The second class constraints can be used to define the Dirac brackets. The decomposition of φ_2, φ_3 into first class and second class components, however, is nontrivial and remains an open question. In this paper, we attempt to calculate the ordinary Poisson bracket between the currents which can be thought of as a first step in the complete evaluation of the algebra of the transition matrices.

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We can calculate the desired $\{\overset{1}{\hat{J}}_1(\sigma,t_1),\overset{2}{\hat{J}}_1(\sigma',t_2)\}$ bracket using (4.5), the relations (6.21) as well as (6.23). Without giving the tedious technical details, we note that the form of this bracket takes the closed form

$$
\begin{aligned}
&\left\{\overset{1}{\hat{J}}_1(\sigma,t_1),\overset{2}{\hat{J}}_1(\sigma',t_2)\right\} = \\
&= \left(\alpha\left[\Omega_P,\overset{(1)}{\hat{J}}_1(\sigma,t_1)\right]+\beta\left[\Omega_P,\overset{(2)}{\hat{J}}_1(\sigma,t_2)\right]+\gamma\left[\Omega_H,\overset{(1)}{\hat{J}}_1(\sigma,t_1)+\overset{(2)}{\hat{J}}_1(\sigma,t_2)\right]\right)\delta(\sigma-\sigma')+ \\
&\quad +\xi_2\left\{\overset{1}{\hat{J}}_1(\sigma,t_1),\overset{2}{\varphi}_2(\sigma')\right\}+\chi_2\left\{\overset{1}{\hat{J}}_1(\sigma,t_1),\overset{2}{\varphi}_3(\sigma')\right\}+ \\
&\quad +\xi_1\left\{\overset{1}{\varphi}_2(\sigma'),\overset{2}{\hat{J}}_1(\sigma,t_1)\right\}+\chi_1\left\{\overset{1}{\varphi}_3(\sigma'),\overset{2}{\hat{J}}_1(\sigma,t_1)\right\}+\Lambda\partial_\sigma\delta(\sigma-\sigma')\,,
\end{aligned}
\tag{6.31}
$$

where $\alpha,\beta,\gamma,\xi_{1,2}$ and $\chi_{1,2}$ are functions of the spectral parameters t_1,t_2 defined as

$$
\begin{aligned}
\alpha &= \frac{(B_2)^2}{A_1B_2-A_2B_1}\,, \qquad \beta=\alpha\left(\frac{B_1}{B_2}\right)^2, \qquad \gamma=\alpha\frac{B_1}{B_2}\,, \\
\xi_2 &= \gamma C_2\,, \qquad \chi_2=\gamma D_2\,, \qquad \xi_1=-\gamma C_1\,, \qquad \chi_1=-\gamma D_1\,,
\end{aligned}
\tag{6.32}
$$

$$
\begin{aligned}
\Lambda &= (A_1B_2+A_2B_1)\,\Omega_P+(\xi_2D_1+\chi_1C_2)\,\Omega_{Q^{21}}+ \\
&\quad +(\xi_1D_2+\chi_2C_1)\,\Omega_{Q^{12}}\,,
\end{aligned}
\tag{6.33}
$$

with

$$
A_i=\frac{1+t_i^2}{1-t_i^2},\; B_i=\frac{2t_i}{1-t_i^2},\; C_i=\sqrt{\frac{1+t_i}{1-t_i}},\; D_i=\sqrt{\frac{1-t_i}{1+t_i}}\,. \tag{6.34}
$$

In the bosonic limit, i.e. when one sets all the fermions to zero, this reduces to the result of [36]. In the presence of the fermions one has additional terms depending on ξ_i,χ_i as well as non-ultralocal terms involving Λ. The terms depending on ξ_i,χ_i are there primarily because we have not yet separated the constraints into (κ symmetry generating) first class constraints and second class constraints. Once this is done and the second class constraints are used to define Dirac brackets, then, in the Dirac bracket of the currents, such terms will be absent and the algebra will have a closed structure. The κ symmetry can, in fact, be fixed in the action as has been suggested in [28, 29, 43, 44]. It is also interesting to understand the meaning of the algebra of the transition matrices on the SYM side. This is a topic presently under study.

The presence of the Λ dependent non-ultralocal terms, on the other hand, leads to a different issue. Analogous to the bosonic case considered in [36], one has to deal with the fact that the non-ultralocal term in (6.23) will lead to an ambiguity in the calculation of the bracket between the transition matrix (6.2). There have been several methods proposed to regularize this ambiguity for the PCM (principal chiral model) and other models. They are based either on regularizing [45] the Poisson bracket between the currents (6.31) by defining symmetrized “weak” Poisson brackets, or by regularizing the transition matrices by introducing a ”retarded” monodromy matrix [46]. Another method due to Faddeev and Reshetikhin [47] views the non-ultralocal terms as a consequence of false vacuum in the classical limit and correspondingly modify the vacuum structure of the theory.

7. Summary and discussions

In this paper, we have constructed nonlocal conserved currents for the superstring in $AdS_5 \times S_5$ background, investigated the hamiltonian structure and presented the algebra of the transition matrices. It is noted that the evolution of the superstring in $AdS_5 \times S_5$ is most conveniently described in Green-Schwarz formalism and the action is expressed as a nonlinear σ model on the coset $\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)}$. It is pointed out that superalgebra admits a Z_4 automorphism which determines the maximal subgroup as $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. The model naturally contains a current satisfying the zero curvature condition.

We introduce a constant spectral parameter which enables us to define a family of new currents such that the vanishing of curvature of these new currents leads to the equations of motion as well as the flatness of the original current. These constructed currents, in our formulation, are invariant under definite transformation rules which are generalizations of the Z_4 automorphism of the superalgebra. This is completely parallel to the bosonic sigma models where this generalized automorphism is used in the construction of the monodromy matrix and the nonlocal currents.

It is more appropriate to adopt the hamiltonian formalism to compute the Poisson bracket algebra of the transition matrices (which can yield the algebra of the charges). The hamiltonian analysis is carried out in a frame work where the spatial component of the current is identified as dynamical variables and its time component gets determined from the zero curvature condition. Subsequently, the canonical momenta are identified. Next, the constrained hamiltonian analysis is carried out to identify the constraints and the algebra of the constraints is presented. When we compute the Poisson bracket between the new currents (defined with the spectral parameter) there are additional terms which include Schwinger term (derivative of the δ-function). As a cross check, if we set all fermionic coordinates to zero, we are able to recover the algebra of such currents derived earlier for purely bosonic σ-models. However, the presence of the κ symmetry makes it difficult to separate the constraints into first and second class ones and construct the Dirac brackets. It is necessary, therefore, to fix the κ symmetry, this is presently under study.

The presence of the non-ultralocal term in the algebra of the current leads to ambiguities in the computation of brackets between the transition matrices. There already exist proposals to regularize such terms as we have discused earlier. The essential difference between the bosonic sigma model coupled to gravity and the present theory is that here we have a constant spectral parameter. We note, however, that in the present theory we do have a $\mathbb{Z}_4^\infty$ invariance (4.10) that leads to a symmetry under $t \longrightarrow \frac{1}{t}$ much like in the bosonic case. This naturally suggests that one way to regularize the non-ultralocal terms is to assume that the spectral parameter is a local function satisfying (4.8) (which would correspond to having a dilaton field in the theory). This would naturally regularize [36] the ambiguity in (6.2) and only at the end of the calculation one should take the limit of a constant spectral parameter. This is an interesting possibility that needs further work and is under investigation.

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A. Useful identities

We present here some relations that have proved useful in the text.

$$
\begin{aligned}
\left[\Omega_H, \overset{1,2}{P}\right] &= -\left[\Omega_P, \overset{2,1}{P}\right], & \left[\Omega_P, \overset{1}{H}\right] &= -\left[\Omega_P, \overset{2}{H}\right],\\
\left[\Omega_{Q^{12}}, \overset{2}{Q}{}^2\right] &= -\left[\Omega_P, \overset{1}{Q}{}^2\right], & \left[\Omega_{Q^{21}}, \overset{2}{Q}{}^1\right] &= -\left[\Omega_P, \overset{1}{Q}{}^1\right],\\
\left[\Omega_{Q^{21}}, \overset{1}{Q}{}^2\right] &= \left[\Omega_P, \overset{2}{Q}{}^2\right], & \left[\Omega_{Q^{12}}, \overset{1}{Q}{}^1\right] &= \left[\Omega_P, \overset{2}{Q}{}^1\right],\\
\left[\Omega_{Q^{12}}, \overset{2}{Q}{}^1\right] &= -\left[\Omega_H, \overset{1}{Q}{}^1\right], & \left[\Omega_{Q^{21}}, \overset{2}{Q}{}^2\right] &= -\left[\Omega_H, \overset{1}{Q}{}^2\right],\\
\left[\Omega_{Q^{12}}, \overset{1}{Q}{}^2\right] &= \left[\Omega_H, \overset{2}{Q}{}^2\right], & \left[\Omega_{Q^{21}}, \overset{1}{Q}{}^1\right] &= \left[\Omega_H, \overset{2}{Q}{}^1\right],\\
\left[\Omega_{Q^{12}}, \overset{1}{H}\right] &= -\left[\Omega_{Q^{12}}, \overset{2}{H}\right], & \left[\Omega_{Q^{21}}, \overset{1}{H}\right] &= -\left[\Omega_{Q^{21}}, \overset{2}{H}\right],\\
\left[\Omega_{Q^{12}}, \overset{1}{P}\right] &= -\left[\Omega_{Q^{21}}, \overset{2}{P}\right], & \left[\Omega_{Q^{12}}, \overset{2}{P}\right] &= -\left[\Omega_{Q^{21}}, \overset{1}{P}\right],\\
\left[\Omega_H, \overset{1}{H}\right] &= -\left[\Omega_H, \overset{2}{H}\right]. & &
\end{aligned}
\tag{A.1}
$$

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